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Control of Formations under Persistent Disturbances

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***Abstract*— We study the distributed control of autonomous second order agents under persistent disturbances. We show that the usual averaging rule for convergence to formation is only able to reject constant disturbances that are identical for each agent. We also prove that using a distributed dynamic compensation law the system can be made to converge to formation under constant perturbations of the control input even when the perturbations are different for each agent.**

Keywords: formation stability, decentralized control, disturbance rejection, dynamic compensation, graph Laplacian.

I. Introduction

There is by now a standard approach to the distributed control of autonomous agents in order to achieve a predetermined formation or a consensus objective. The feedback law used was originally motivated by the organized motions of birds in flocks and fish in schools ([1]) and as a model for self driven particles [2]. The model was first used for the control of vehicle formations in [3], [4] and latter studied by many others (see [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18] and the references therein).

In this paper we focus our attention on proving under which conditions the feedback law can stabilize the system to formation in the presence of certain persistent disturbances. Roughly speaking, the feedback rule simply calculates an (weighted) average of the relative errors between the positions and velocities of an agent and its neighbors and the desired relative position and velocities of the corresponding formation objective. The "neighbors" refers to a communication digraph and the weights can be either assumed to come from cost on the communication links (edges of the graph) or from some feedback gains in the control law.

The paper is organized as follows. In Section II present the model and several formal definitions. We summarize some facts from graph theory in Section III. The main results are proven in Sections IV and V. We illustrate the results in Section VI and summarize the main points in Section VII.

II. Model

We assume we are given N agents (or vehicles) with the same dynamics

$$\dot{x}_i = A_{veh}x_i + B_{veh}u_i \qquad i = 1 \dots N \quad x_i \in \mathbb{R}^{2n} \tag{1}$$

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where the entries of x_i represent n configuration variables for agent i and their derivatives, and the u_i represent control inputs. The matrices A_{veh} and B_{veh} are of the form

$$A_{veh} = \begin{pmatrix} 0 & 1 & 0 & 0 & & & \cdots & 0 \\ 0 & a_{22} & 0 & a_{24} & 0 & a_{26} & \cdots & a_{2(2n)} \\ 0 & 0 & 0 & 1 & 0 & & \cdots & 0 \\ 0 & a_{42} & 0 & a_{44} & 0 & a_{46} & \cdots & a_{4(2n)} \\ \vdots & & & \vdots & & & & \vdots \end{pmatrix}$$

$$B_{veh} = \begin{pmatrix} 0 & 0 & 0 & \cdots & \\ 1 & 0 & 0 & \cdots & \\ 0 & 0 & 0 & \cdots & \\ 0 & 1 & 0 & \cdots & \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \\ \vdots & & & & \end{pmatrix}. \tag{2}$$

The form of the odd-numbered rows of A_{veh} and B_{veh} is determined by the fact that the even-numbered coordinates represent the velocities of the (previous) odd-numbered coordinates and that the controls affect the accelerations. The zeros in the odd-numbered columns of A_{veh} are necessary for the vehicles to converge to formation (see [6] Proposition 3.1 and [19] Proposition 4.2). The entries of the form $a_{(2k)(2k)}$ affect the acceleration of the formation as a whole: when negative the agents achieve formation and stop, when zero the agents achieve formation while drifting, and when positive the agents achieve formation but the formation as a whole accelerates ([7]). The other entries are related to a rotational movement of the formation ([20]).

We will refer to the odd-numbered entries of $x = (x_1, \dots, x_N)^T$ as *position-like* variables and to the even-numbered entries as *velocity-like* variables. We will use the notation $x_p = ((x_p)_1, \dots, (x_p)_N)^T$, $x_v = ((x_v)_1, \dots, (x_v)_N)^T$ to denote the vectors of position-like and velocity-like variables, so $x = x_p \otimes \begin{pmatrix}1\\0\end{pmatrix} + x_v \otimes \begin{pmatrix}0\\1\end{pmatrix}$ (where $\otimes$ denotes the Kronecker product).

Definition 2.1: A **formation** is a vector $h = h_p \otimes \begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{R}^{2nN}$ (where $\otimes$ denotes the Kronecker product). The N agents are **in formation** h at time t if there are vectors $q, w \in \mathbb{R}^n$ such that $(x_p)_i(t) - (h_p)_i = q$ and $(x_v)_i(t) = w$, for $i = 1 \dots N$. The vehicles **converge to formation** h if there exist $\mathbb{R}^n$-valued functions $q(\cdot), w(\cdot)$ such that $(x_p)_i(t) - (h_p)_i - q(t) \to 0$ and $(x_v)_i(t) - w(t) \to 0$, as $t \to \infty$, for $i = 1 \dots N$ (where x_p and x_v are as indicated above).

Fig. 1 illustrates the interpretation of the various vectors in the definition.

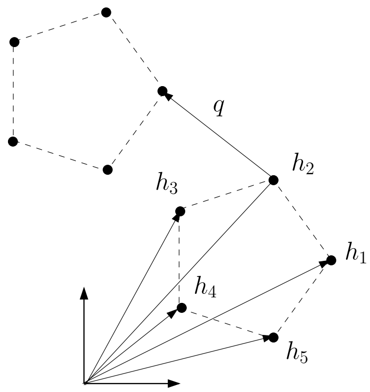

Fig. 1. Five agents in pentagon formation

Let $h = h_p \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^{2nN}$ and let $\mathbf{1}_N$ denote the all ones vector of size N. Notice that $x - h = \mathbf{1}_N \otimes \gamma$ is equivalent to $(x_p)_i - (h_p)_i = q$ and $(x_v)_i = w$ where $\gamma = q \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + w \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

To complete the model we are also given a digraph Γ which captures the communication links between agents (see Section III). Each vertex represents an agent and there is a directed edge from one vertex to another if there is a communication link sending information from the first agent to the second. The second agent uses this information in a feedback formula to adjust its own state. We say that the first agent is a neighbor of the second. For each agent i, $\mathbb{J}_i$ denotes the set of its neighbors. The decentralized nature of the control is encoded into the fact that controls u_i are functions of $x_j - x_i$ and $h_j - h_i$ for each $j \in \mathbb{J}_i$.

The standard model analyzed in the literature ([4], [7], [21]) uses a linear feedback of the output functions z_i computed from an average of the relative displacements (and velocities) of the neighboring agents. Furthermore, to allow for "leaders" (agents which only send information to other agents but do not receive, so the other agents must adjust their motion to the leader) the output is rewritten as

$$z_i = \begin{cases} \frac{1}{|\mathbb{J}_i|} \sum_{j \in \mathbb{J}_i} ((x_i - h_i) - (x_j - h_j)) & \text{if } |\mathbb{J}_i| \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, N$. As a result, the corresponding output vector z can be written as $z = L(x - h)$ where $L = L_\Gamma \otimes I_{2n}$ and L_Γ is the (directed) Laplacian matrix of the communication graph Γ (see Section III).

Collecting the equations for all the vehicles into a single system we obtain

$$\dot{x} = Ax + Bu \quad (3)$$

$$z = L(x - h) \quad (4)$$

with $A = I_N \otimes A_{veh}$, $B = I_N \otimes B_{veh}$. The problem of the existence of feedback matrices F such that the solutions to

$$\dot{x} = Ax + BFL(x - h)$$

converge to formation h, has been well studied. The problem is solvable if and only if the communication graph admits a (rooted) directed spanning tree ([7], [21]). From now on we will assume this is the case.

The focus of this paper is to study the effect of various disturbances on this model. We recast the problem as a classical output stabilization problem ([22]) and prove two results:

1) **The disturbance decoupling problem is solvable if and only if the disturbances are constant (zero velocity) an equal for all agents.**
2) **The regulator problem with internal stability is solvable in the presence of constant feedback disturbances (even if different for each agent).**

III. Graph Theory

For general graph theoretic references we refer the reader to [23]. A *directed graph* or *digraph* Γ consists of a finite set $\mathcal{V}$ of *vertices* and a set $\mathcal{E} \subseteq V \times V$ (the directed *edges*). We will assume that the digraph has no loops, that is $(x, x) \notin \mathcal{E}$ for any x. A graph is undirected if for all $x, y \in \mathcal{V}$, $(x, y) \in E$ implies $(y, x) \in E$.

Let Γ denote a digraph with vertex set $\mathcal{V} = \{i \colon i = 1, \ldots, N\}$ and edge set $\mathcal{E}$. The *adjacency matrix* of Γ is the $N \times N$ matrix Q with entries

$$q_{ij} = \begin{cases} 1 & \text{if } (j, i) \in \mathcal{E}, \\ 0 & \text{otherwise} \end{cases} \qquad (i, j \in \mathcal{V}).$$

When Γ is undirected, the matrix Q is symmetric. The *in-degree matrix* of Γ is the diagonal $N \times N$ matrix D with diagonal entries

$$d_{ii} = |\{j \in \mathcal{V} \,:\, (j, i) \in \mathcal{E}\}| \qquad (i \in \mathcal{V}).$$

The *directed* Laplacian of Γ is the matrix defined by ([19])

$$L_\Gamma = D^+(D - Q),$$

where D^+ is the (Moore-Penrose) pseudoinverse of D. This is slightly different from the standard matrix Laplacian (see [20]). A key property of L_Γ is that zero is an eigenvalue of L_Γ and the all ones vector $\mathbf{1}_N$ is an associated eigenvector (but in general there could be others [24]). All the eigenvalues of L_Γ lie in the circle of radius 1 centered at the point $1 + 0i$ in the complex plane. In particular, all nonzero eignevalues have positive real part (for additional properties see [7]).

Definition 3.1: A *rooted directed tree* is a digraph T with the following properties:

- T has no cycles.
- There exists a vertex v (the root) such that there is a (directed) path from v to every other vertex in T.

The following result was proved in [7] and [21].

Proposition 3.2: Let G denote a (loopless) digraph. Then, zero is an eigenvalue of algebraic multiplicity one for the directed Laplacian L_Γ if and only if Γ has a rooted directed spanning tree.

In what follows we will only be interested in the case when zero is an eigenvalue of multiplicity one of Γ.

IV. Disturbance Rejection

We consider first the disturbance decoupling problem. More precisely, we investigate the model

$$\begin{aligned} \dot{x} &= Ax + Bu + Sq && (5) \\ z &= L(x-h) && (6) \end{aligned}$$

where S is constant. We will assume that the disturbances are piecewise continuous, but otherwise arbitrary. The main question is whether there exist matrices F such that setting $u = Fz$ in Eq. (5-6) guarantees that $z(\cdot)$ is the same for any disturbance $q(\cdot)$. We refer to this as the Disturbance Rejection Problem (DRP). Using the explicit form for the solution of the system we have,

$$z(t) = L\left(\int_0^t e^{(t-s)(A+BFL)}(Sq(s) - BFLh)\, ds - h\right)$$

The problem is then that of determining if there exists F such that for any function $q(\cdot)$

$$z(t) = L\left(\int_0^t e^{(t-s)(A+BFL)}Sq(s)\, ds\right) = 0. \quad (7)$$

Since the null space of L_G consists solely of vectors of the form $c\mathbf{1}_N$, the null space of L consists of vectors of the form $\mathbf{1}_N \otimes \alpha$, where $\alpha \in \mathbb{R}^{2n}$. Equation 7 is then equivalent to

$$\int_0^t e^{(t-s)(A+BFL)}Sq(s)\, ds = \mathbf{1}_N \otimes \alpha(t) \quad (8)$$

for some $\mathbb{R}^{2n}$-valued continuous function $\alpha(\cdot)$. To facilitate the explanation we introduce some additional notation. Given an $p \times p$ matrix M and a vector subspace $\mathcal{T}$ of $\mathbb{R}^p$ we denote by $\langle M|\mathcal{T}\rangle$ the subspace

$$\mathcal{T} + M\mathcal{T} + \cdots + M^{p-1}\mathcal{T}.$$

(This is the controllable subspace of (M, T) for any matrix with column space $\mathcal{T}$.) With this notation (8) holds if and only if $\langle A + BFL|\mathcal{S}\rangle \subseteq \mathcal{S}$ where $\mathcal{S}$ is the column space of S. In particular $\mathcal{S} \subseteq \{\mathbf{1}_N \otimes \alpha \colon \alpha \in \mathbb{R}^{2n}\} = \text{Null}(L)$. On the other hand, since $L(\mathbf{1}_N \otimes \alpha) = 0$, we get $(A+BFL)^k(\mathbf{1}_N \otimes \alpha) = A^k(\mathbf{1}_N \otimes \alpha))$. Recalling that $A = I \otimes A_{veh}$, we see that $A^k(\mathbf{1}_N \otimes \alpha)) = I \otimes A^k_{veh}\alpha$ which is again in the Null space of L. We have then proved the following.

Proposition 4.1: The DRP is solvable if and only if $\mathcal{S} = \text{Null}(L)$.

To paraphrase, the only disturbances that can be decoupled are those that are exactly the same for each agent. One such example, in case the agents are flying vehicles, would be a wind that could vary over time. The wind could include sudden gusts as long as all vehicles are affected equally. The output would ignore this type of inputs, meaning that the formation will not be perturbed. Given the nature of the feedback law (neighboring data averaging) this result is rather intuitive.

V. Output Stabilization

We now consider the problem of asymptotic stabilization (convergence to formation) in the presence of persistent control errors. We consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu && (9) \\ z &= L(x-h) && (10) \end{aligned}$$

but this time we look for matrices F such that if we set $u = Fz + v$ for an arbitrary constant v the system will still converge to formation. In fact, the theory can be applied as long as the disturbance v satisfies a known time-invariant linear differential equation. This becomes clear in the approach because we will expand the system to include the disturbances as state variables. Here we just assume they are constant and so they satisfy the equation $\dot{v} = 0$. We will expand the system to include the compensator $\dot{\tilde{x}} = z$, essentially integrating the average output error.

We first observe that given the structure of A_{veh} we can dispose of h altogether. Indeed, notice that $Ah = 0$. Therefore we can write the system as follows

$$\dot{x} = Ax + Bu = A(x-h) + Bu$$

$$z = L(x-h)$$

Since h is constant we can change variables to $\tilde{x} = x - h$ and we obtain the same dynamics and output as before but with $h = 0$. **To avoid more cumbersome notation we will heretofore assume that $h = 0$.**

To cast the problem in classical terms we expand the system to include v as a state variable with appropriately modified dynamics. We use the subscript e to denote the expanded objects, so

$$x_e = \begin{pmatrix} x \\ v \end{pmatrix} \quad A_e = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \quad B_e = \begin{pmatrix} B \\ 0 \end{pmatrix}$$

$$L_e = \begin{pmatrix} L & 0 \end{pmatrix}$$

The extended system is as follows.

$$\begin{aligned} \dot{x}_e &= A_e x_e + B_e u && (11) \\ z &= L_e x_e && (12) \end{aligned}$$

The problem is now to find F, (if possible) such that under the feedback $u = Fz$, for any initial condition $x_e(0)$, we get $z \to 0$ as $t \to \infty$. Notice that the disturbance enters the equations through the submatrix B within A_e. This is a special case of the Regulator Problem with Internal Stabilization (RPIS) (see [22], Chap. 7). As such we will show that it is indeed solvable.

Notice that in (11)-(12) $x_e \in \mathbb{R}^{2nN+nN}$, $u \in \mathbb{R}^{nN}$, and $z \in \mathbb{R}^{2nN}$. The matrices and their blocks have the corresponding dimensions. We denote by $\mathcal{B}_e$ the space spanned by the columns of B_e, and by $\mathcal{N}$ the space

$$\mathcal{N} = \bigcap_{i=0}^{3nN-1} \text{Null}(L_e A_e^i).$$

($\mathcal{N}$ is the unobservable space.) Finally, we denote by $\mathcal{X}^+(A_e)$ the unstable subspace of A_e, that is, the null space of $p^+(A_e)$, where $p^+(\lambda)$ is the unstable part of the minimal polynomial of A_e. We will use the following result.

Theorem 5.1 (Wonham [22]): RPIS is solvable if and only if there exists a subspace $\mathcal{V}$ of $\mathbb{R}^{3nN}$ such that

$$\mathcal{V} \subset \text{Null}(L_e) \cap A^{-1}(\mathcal{V} + \mathcal{B}_e) \quad (13)$$
$$\mathcal{X}^+(A_e) \cap \mathcal{N} + A(\mathcal{V} \cap \mathcal{N}) \subset \mathcal{V} \quad (14)$$
$$\mathcal{V} \cap (\langle A_e|\mathcal{B}_e\rangle + \mathcal{N}) \subset \mathcal{N} \quad (15)$$
$$\mathcal{X}^+(A_e) \subset \langle A_e|\mathcal{B}_e\rangle + \mathcal{V} \quad (16)$$

We now proceed to characterize the relevant spaces above in more detail. Note first that the columns of B_e are the canonical vectors $\mathbf{e}_{2j}$ in $\mathbb{R}^{3nN}$ for $j = 1, \ldots, nN$.

Proposition 5.2: The subspace $\mathcal{V} = \text{Null}(L_e)$ satisfies (13)-(16).

Proof The space $\mathcal{V}$ consists of all vectors of the form

$$\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} \mathbf{1}_N \otimes \alpha \\ v \end{pmatrix}$$

with α and v arbitrary. This follows directly form the shape of L_e and the characterization of the null space of L given earlier. If $x_e \in \mathcal{V}$ then

$$A_e x_e = \begin{pmatrix} Ax + Bv \\ 0 \end{pmatrix} = \begin{pmatrix} A(\mathbf{1}_N \otimes \alpha) + Bv \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} I_N \otimes A_{veh}\alpha + Bv \\ 0 \end{pmatrix}.$$

Therefore $A_e x_e \in \mathcal{V} + \mathcal{B}_e$. This proves (13).

To compute $\mathcal{N}$ observe that $L_e A_e^i = \begin{pmatrix} LA^i & LA^{i-1}B \end{pmatrix}$. Therefore, for $x_e = \begin{pmatrix} x \\ v \end{pmatrix}$ to be in $\mathcal{N}$ we must have, in particular, $Lx = 0$ and $L(Ax + Bv) = 0$. This means that $x = \mathbf{1}_N \otimes \alpha$ and $Ax + Bv = \mathbf{1}_N \otimes \gamma$. Since $A(\mathbf{1}_N \otimes \alpha) = \mathbf{1}_N \otimes A_{veh}\alpha$ we get $L(Ax + Bv) = LBv$. Using the special form of B we then get $v = \mathbf{1}_N \otimes \beta$ with $\beta \in \mathbb{R}^n$. This implies also that $L_e A_e^i x_e = 0$ for all $i \geq 2$. In summary,

$$\mathcal{N} = \left\{ \begin{pmatrix} \mathbf{1}_N \otimes \alpha \\ \mathbf{1}_N \otimes \beta \end{pmatrix} : \alpha \in \mathbb{R}^{2n}, \beta \in \mathbb{R}^n \right\}.$$

Let $x_e \in \mathcal{N}$. Then

$$A_e x_e = \begin{pmatrix} A(\mathbf{1}_N \otimes \alpha) + B(\mathbf{1}_N \otimes \beta) \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{1}_N \otimes A_{veh}\alpha + \mathbf{1}_N \otimes (I_n \otimes \beta) \\ 0 \end{pmatrix}.$$

Which shows that $A_e x_e \in \mathcal{V}$. Together with $\mathcal{N} \subset \mathcal{V}$ the above proves (14). For the next two inclusions we need to compute $\langle A_e|\mathcal{B}_e\rangle$. A direct calculation shows $A_e^k B_e^k = \begin{pmatrix} I_N \otimes A_{veh}^k B_{veh} \\ 0 \end{pmatrix}$ and, since the pair (A_{veh}, B_{veh}) is completely controllable, the space $\langle A_e|\mathcal{B}_e\rangle$ consists of all vectors of the form $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ for $\alpha \in \mathbb{R}^{2nN}$. Then $\langle A_e|\mathcal{B}_e\rangle + \mathcal{N}$ consists of vectors of the form $\begin{pmatrix} \alpha \\ \mathbf{1}_N \otimes \beta \end{pmatrix}$ and so $\mathcal{V} \cap (\langle A_e|\mathcal{B}_e\rangle + \mathcal{N}) = \mathcal{N}$. Therefore (15) holds.

Finally notice that $\langle A_e|\mathcal{B}_e\rangle + \mathcal{V} = \mathbb{R}^{3nN}$ and therefore (16) holds as well, regardless of the entries $a_{(2j)(2k)}$ in the matrix A_{veh}. □

Combined, the last theorem and proposition give the main result.

Proposition 5.3: The RPIS problem for the agent formation problem with constant feedback error is solvable.

The above results, while useful because they allow for an easy check, do not indicate how to compute a stabilizing feedback. However, a more detailed approach used in [22] reduces in this case to finding a feedback law F which stabilizes $A + BFL$ on the observable quotient space. When using a feedback law in block form identical for each agent, that is, for $F = I_N \otimes F_{veh}$, the desired F is obtained when F_{veh} is such that $A_{veh} + \lambda B_{veh} F_{veh}$ is Hurwitz for each nonzero eigenvalue λ of the Laplacian L_G ([7]).

VI. Examples

We illustrate the RPIS results with some numerical simulations representing five autonomous agents moving on a plane ($n = 2$). The entries $a_{(2k)(2k)}$ are set to zero to allow some drift and better illustrate the effects. In all figures below the agents start in a straight line and the goal is for them to arrange themselves in a regular pentagon formation. The overall formation motion path is not planned nor is it tracked. Instead, the shown trajectories are arbitrary results of both the vehicle dynamics needed to achieve the commanded formation and external disturbances. In Figs. 2 and 3 all even rows in A_{veh} are set to zero. In Fig. 2 the standard averaging feedback law is used but a constant disturbance is added to the feedback loop as explained in the RPIS problem. The final positions of the agents are indicated by the colored dots. The color traces indicate the path of each agent. The pentagon lines are included to make it easier to visualize the relative positions of the agents. The agents fail to achieve the regular pentagon formation as is clearly visible. As discussed above, this is because only the averaging feedback law is used, but no integral compensator.

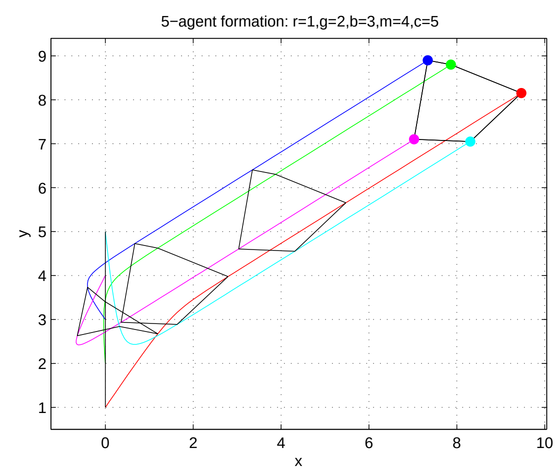

Fig. 2. Agents do not achieve formation with feedback disturbance

In Fig. 3 the compensator is turned on about a third of the way through the motion to illustrate its effect. Initially the agents settle in an irregular pentagon formation. After the compensator is turned on there is some intermediate transient behavior and then the regular pentagon formation is achieved.

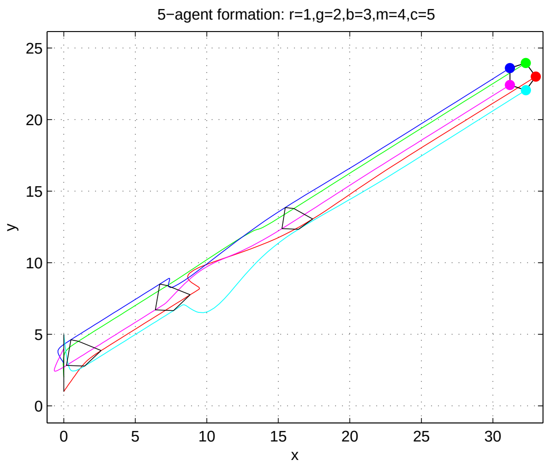

Fig. 3. Convergence with compensator

In Figs. 4 and 5 we present a similar situation while the agents perform a circular motion. Here we set $a_{24} = -a_{42} \neq 0$. In Fig. 4 there is no integral compensation and the agents converge to a distorted formation. In Fig. 5 about half way through the motion the agents have again settled in an irregular pentagon pattern. At that point the compensator is turned on and, after an initial transient, the agents then achieve the desired formation. Because of the effect of the disturbances the resulting motion is not perfectly circular. The compensator cancels the effect of the disturbance on the formation but not on the absolute positions of the agents. Such absolute motions reside in the unobservable space of the model.

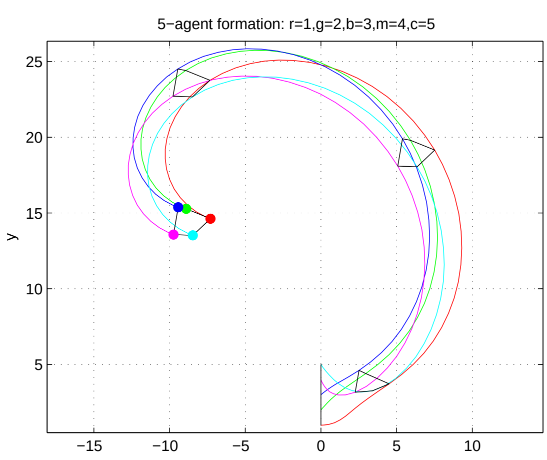

Fig. 4. Feedback disturbance in circular motion

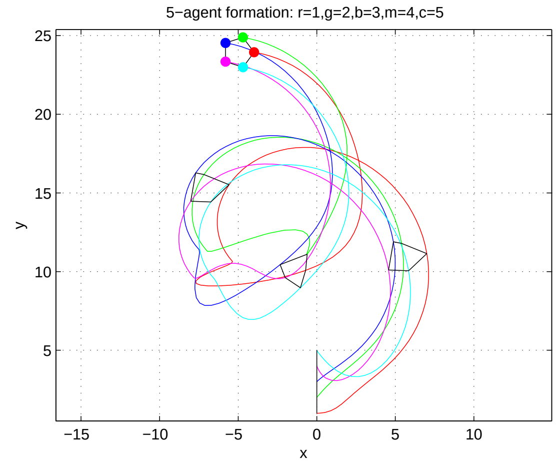

Fig. 5. Convergence while in circular motion

VII. Concluding Remarks

We showed that the standard formation problem discussed in the literature is disturbance decoupled as long as the disturbance is constant across agents. We also showed that by using a simple compensator the system can cancel out constant perturbations in the feedback control loop. It should be pointed out that the compensator is also distributed in the sense that each agent need only know its own relative errors in order to generate the compensating feedback.

The underlying theory is more general than illustrated in the paper, as far as the types of disturbances that can be handled. More complicated disturbances will require correspondingly more complex dynamics in the compensator. The input to the compensator will still be the output error $z = L(x - h)$.

References

[1] A. Okubo, "Dynamical aspects of animal grouping: swarms, schools, flocks and herds," *Advances in Biophysics*, vol. 22, pp. 1–94, 1986.

[2] T. Vicsek, A. Czirok, E. B. Jacob, I. Cohen, and O. Schochet, "Novel type of phase transitions in a system of self-driven particles," *Physical Review Letters*, vol. 75, pp. 1226–1229, 1995.

[3] J. A. Fax, "Optimal and cooperative control of vehicle formations," Ph.D. dissertation, California Institute of Technology, 2001.

[4] J. Fax and R. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1465–1476, 2003.

[5] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 998–1001, June 2003.

[6] G. Lafferriere, J. Caughman, and A. Williams, "Graph theoretic methods in the stability of vehicle formations," *ACC2004*, pp. 3729–3724, July 2004.

[7] G. Lafferriere, A. Williams, J. Caughman, and J. J. P. Veerman, "Decentralized control of vehicle formations," *Systems and Control Letters*, vol. 54, pp. 899–910, March 2005.

[8] J. Lawton, R. Beard, and B. Young, "A decentralized approach to formation maneuvers," *IEEE Transactions on Robotics and Automation*, vol. 19, no. 6, pp. 933–941, December 2003.

[9] N. Leonard and E. Fiorelli, "Virtual leaders, artificial potentials and coordinated control of groups," *Proceedings of IEEE Conference on Decision and Control*, pp. 2968–2973, 2001.

[10] N. Leonard and P. Ogren, "Obstacle avoidance in formation," *IEEE ICRA*, pp. 2492 – 2497, September 2003.

[11] W. Ren and R. Beard, "A decentralized scheme for spacecraft formation flying via the virtual structure approach," *AIAA Journal of Guidance, Control and Dynamics*, vol. 27, no. 1, pp. 73–82, January 2004.

[12] A. Sparks, "Special issue on control of satellite formations," *International Journal of Robust and Nonlinear Control*, vol. 12(2-3), 2002.

[13] H. Tanner, A. Jadbabaie, and G. Pappas, "Stable flocking of mobile agents, part I: Fixed topology," in *Proc. IEEE Conference on Decision and Control*, December 2003, pp. 2010–2015.

[14] ——, "Stable flocking of mobile agents, part II: Dynamic topology," in *Proc. IEEE Conference on Decision and Control*, December 2003, pp. 2016–2021.

[15] ——, "Flocking in fixed and switching networks," *Automatica*, Submitted July 2003.

[16] A. Williams, S. Glavǎski, and T. Samad, "Formations of formations: Hierarchy and stability," *ACC2004*, pp. 2992–2997, July 2004.

[17] R. Olfati-Saber and R. Murray, "Agreement problems in networks with directed graphs and switching topology," in *Proc. IEEE Conference on Decision and Control*, December 2003, pp. 4126–4132.

[18] ——, "Consensus protocols for networks of dynamic agents," in *Proc. of the American Control Conference*, July 2003, pp. 951–956.

[19] J. J. P. Veerman, G. Lafferriere, J. S. Caughman, and A. Williams, "Flocks and formations," *J. Stat. Phys.*, vol. 121, no. 516, pp. 901–936, 2005.

[20] A. Williams, G. Lafferriere, and J. Veerman, "Stable motions of vehicle formations," *Proc. of 44th IEEE CDC and ECC '05*, 2005.

[21] W. Ren and R. Beard, "Consensus seeking in multi-agent systems using dynamically changing interaction topologies," *IEEE Trans. Automat. Control*, vol. 50, no. 5, pp. 655–661, 2005.

[22] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 3rd ed. Springer-Verlag, 1985.

[23] D. B. West, *Introduction to Graph Theory*. London: Prentice Hall, 2001.

[24] J. S. Caughman and J. J. P. Veerman, "Kernels of directed graph laplacians," *Electr. J. Comb.*, vol. 1,R39, no. 13, 2006.